Phase and polarization synchronization in vectorial oscillators

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In recent years a considerable interest have been attracted by the networks of coupled nonlinear oscillators¹, since they represent a prototype model for a huge variety of self-organizing systems in physics, chemistry, biology and social science. We have studied the synchronization properties of ensemble of $N$ mean-field coupled non-identical cycle limit oscillators, taking into account the polarization degree of freedom (Vectorial Oscillators). This finds practical applications e.g. when modeling laser oscillators laying in structure not polarization preserving (e.g. arrays of Vertical-Cavity Surface Emitting Lasers in photonics). Two sources of disorder are introduced into the system to prevent synchronization: different natural oscillation frequencies and angles (the latter fixed by complex forcing terms, reflecting the symmetry properties of the environment). The equations set for the time evolution of the phase $\phi$ and the (linear) polarization angle $\psi$ of the $j_{th}$ vectorial oscillator are:

\[
\psi_j = a \sin(\psi_j - \delta_j) + \frac{c}{N} \sum_{k=1}^{N} \sin \Psi_{kj} \cos \Phi_{kj}, 
\]

\[
\dot{\phi}_j = \Omega_j + b \cos(\psi_j - \delta_j) + \frac{c}{N} \sum_{k=1}^{N} \sin \Phi_{kj} \cos \Psi_{kj}, 
\]

where $\Psi_{kj} \equiv (\psi_k - \psi_j)/2$, $\Phi_{kj} \equiv (\phi_k - \phi_j)/2$, $\omega_j$ are the natural oscillation frequencies, $\delta_j$ are the natural oscillations angles, $a$ and $b$ are amplitude and phase anisotropy terms, and $c$ is the (mean-field) coupling extent. This model reverts to the one developed by Kuramoto² for scalar non-identical cycle limit oscillators, when the polarization degree of freedom is disregared.

The polarization and phase synchronization properties are studied through the following order parameters:

\[
\eta \exp(i\chi) = \frac{1}{N} \sum_{k=1}^{N} \exp(i\psi_k/2), 
\]

\[
\rho \exp(i\theta) = \frac{1}{N} \sum_{k=1}^{N} \exp(i\phi_k/2). 
\]

$\eta (\rho) \to 1$ means polarization (phase) complete synchrony.

Increasing the coupling, no polarization order enhancement is possible until the phases start to synchronize, because the phase disorder destroys the interaction among the polarization variables. For strong natural angle disorder, the phases synchronize first, and polarization synchrony takes place at a higher coupling level, through a partial de-synchronization of the phases, as shown in Fig. 1. The degree of de-synchronization depends on the phase anisotropy term $b$. We have developed an approximated analytical theory to estimate the phase and polarization order parameters (solid line in Fig. 1). For weak natural angle disorder, the two transitions merge in a unique process to full synchrony, and we have provided the critical coupling for its onset.

\[\text{Figura 1. Amplitude of the polarization order parameter $\eta$ (upper panel) and the phase order parameter $\rho$ (lower panel) as function of the coupling strength for $a = -1$ and several values of $b$: $b = 0$ (1), $b = 1.21$ (2), $b = 2.4$ (3). Dots correspond to numerical integration of Eqs. (1) and (2) with $N=10000$ and a Gaussian distribution for the natural frequencies, (standard deviation $\sigma_\Omega = 10^{-2}$), and $\rho(\delta) = \Delta^2 + \Delta^2$, with $\Delta = 3 (\sigma_\psi = 2.4)$ for the natural angles. Solid lines correspond to the theoretical prediction. The thick line with star-markers in the upper panel represents the numerical evaluation of the coupling term in the polarization Eq. (1).}\]

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