Collective modes of coupled phase oscillators with delayed coupling

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We study the effects of delayed coupling on timing and pattern formation in spatially extended systems of dynamic oscillators. Starting from a discrete lattice of coupled oscillators, we derive a generic continuum theory for collective modes of long wavelength[1].

The discrete lattice of coupled oscillators can be described by an evolution equation for the phase $\theta_{\mathbf{i}}$ of the oscillator $\mathbf{i}$, coupled to its nearest neighbors $\mathbf{j}$:

$$\frac{d\theta_{\mathbf{i}}(t)}{dt} = \omega_{\mathbf{i}} + \frac{\varepsilon_{\mathbf{i}}}{2d} \sum_{|\mathbf{j}-\mathbf{i}|=1} h(\theta_{\mathbf{j}}(t-\tau) - \theta_{\mathbf{i}}(t)) \ , \qquad (1)$$

where $\omega_{\mathbf{i}}$ is the intrinsic frequency of the oscillator $\mathbf{i}$, $\varepsilon_{\mathbf{i}}$ denotes the coupling strength of this oscillator to its neighbors, and $\tau > 0$ is a time delay in the coupling. The coupling is described by the 2π-periodic function h. Introducing the continuum phase $\theta(x,t)$ and functions $\omega(x)$ and $\varepsilon(x)$ we show that the collective modes of long wavelength of Eq. (1) can be described by:

$$\frac{\partial\theta(x,t)}{\partial t} = \omega(x) + \varepsilon(x) h(\theta(x,t-\tau) - \theta(x,t)) \qquad (2)$$

$$+\frac{\varepsilon(x)a^2}{2} h''(\theta(x,t-\tau) - \theta(x,t)) \left(\frac{\partial\theta(x,t-\tau)}{\partial x}\right)^2$$

$$+\frac{\varepsilon(x)a^2}{2} h'(\theta(x,t-\tau) - \theta(x,t)) \frac{\partial^2\theta(x,t-\tau)}{\partial x^2} \ ,$$

where the prime denotes the derivative of h with respect to its argument.

We use this approach to study spatial phase profiles of cellular oscillators in the segmentation clock, a dynamic patterning system of vertebrate embryos[2,3]. Collective wave patterns result from the interplay of coupling delays and moving boundary conditions[4,5]. We show that the phase profiles of collective modes depend on coupling delays, and derive experimental features of the segmentation clock starting from a very general framework.

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