Threshold phenomenon in the *Countdown* game

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In combinatorial optimization problems, a large amount of literature points out to the occurrence of a so called threshold phenomenon in the performance of search algorithms: there exists a phase in parameter space where the search algorithm can easily find a solution to the aforementioned combinatorial problem (as the number of available solutions is exponentially large with the system size), and a phase where such solution typically (i.e. almost surely) does not exist[1]. The transition between both phases is sharp in some situations, mimicking in several aspects the phenomenon of a phase transition in statistical physics problems. Some classical problems evidencing such phenomenon include combinatorial problems in random graphs or the satisfaction of (random) boolean clauses, generically gathered under the umbrella of random constraint satisfaction problems (rCSP)[1]. Many of these concrete problems can indeed by interpreted under a statistical physics formalism[2], the general idea being the following: in a combinatorial optimization problem, is some cases one can formalize a cost function to be minimized. In satisfaction problems, this is for instance the number of violated constraints. Within statistical physics of disorders systems, such as in spin glasses, one indeed proceeds in the same manner if the system is studied in the limit of low temperature (in that situation, the system tries to adopt the ground state or minimal energy configuration). Thereby, the cost function of a combinatorial optimization problem can be related to the Hamiltonian of a disordered system at zero temperature (for instance, finding a minimum partition within the so-called partitioning problem is equivalent to finding the ground state of a infinite range Ising spin glass with Mattis-like, antiferromagnetic couplings[3]).

In this work we present a random decision problem, called the *countdown game*, which is inspired in a celebrated british TV quiz show called Countdown (based itself on the French game show *Des chiffres et des lettres*, one of the longest-running game shows in the world, and receiving other names in several countries[5]). This show is separated in several games, one of which incorporates the combinatorial problem of arithmetically combining some numbers to produce another one. Concretely, the contestants must use arithmetic to reach a given target number from six other numbers. Here we formalize a random version of this decision problem and explore its solvability as a function of the parameter space. After defining the game in a similar fashion as random constraint satisfaction problems, we numerically explore the ability of a search algorithm to solve it, and show that this probability drops sharply from typically one to zero (where typically is synonymous to almost surely in the thermodynamic limit). This suggests that that the game evidences the threshold phenomenon. At odds with standard decision problems evidencing sharp algorithmic transitions either in physics or computer science, we show that within this problem such behavior is genuinely of a number-theoretical garment. We take advantage of this fact to characterize the transition, which is of a purely algorithmic nature, in terms of combinatorial number theory. Some discussions regarding possible relations of this problem to more general questions in random group theory, and other alternative approaches are also presented.

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[1] C. Moore and S. Mertens, The Nature of Computation, Oxford University Press (2011).

[2] M. Mezard, G. Parisi, and M. Virasoro, Spin glass theory and beyond (World Scientific, Singapore, 1987).

[3] S. Mertens, A Physicist's Approach to Number Partitioning. Theoret. Comput. Sci. 265, 79-108 (2001).

[4] S. Kirkpatrick and B. Selman, Critical Behavior in the Satisfiability of Random Boolean Expressions,Science 264, 5163 pp. 1297-1301 (1994).

[5] Wikipedia entry for Countdown game show: http://en.wikipedia.org/wiki/Countdown_(game_show)