Low-Dimensional Dynamics of Populations of Pulse-Coupled Oscillators

Ernest Montbrió*, Diego Pazó†
Department of Information and Communication Technologies
Universitat Pompeu Fabra 08018-Barcelona

Clapping audiences, pedestrians in the London's millennium bridge, and flashing fireflies offer spectacular examples of collective synchronization. This pervasive tendency towards synchrony also occurs at the microscopic level, where heart pacemaker cells self-organize their rhythmic activity to initiate the heartbeat. In 1967, Arthur Winfree proposed a mathematical model[1] that successfully replicated this natural phenomenon of self-organization in time, and initiated a prolific research program on collective synchronization. Only a few years after Winfree's seminal paper, Kuramoto proposed a model singularly amenable to mathematical analysis[2], which rapidly became the canonical model to mathematically investigate synchronization phenomena.

Despite their importance and generality, Kuramoto-like models —in which interactions are expressed by phase differences— are approximations of more realistic models such as the Winfree model, and typically their parameters do not have a simple mapping with biologically meaningful parameters. In contrast, the Winfree model incorporates explicit pulse-like interactions and phase response curves (PRCs) that are customarily obtained from experiments or from biologically realistic conductance-based models.

In this contribution we report on the first complete mathematical analysis of the Winfree model. Winfree considered of a large population of pulse-coupled oscillators, and the assumtion of weak coupling permited him to describe the state of each oscillator solely by its phase variable θ:

$$\dot{\theta}_i = \omega_i + Q(\theta_i)\frac{\varepsilon}{N}\sum_{j=1}^{N} P(\theta_j), \qquad (1)$$

where ε controls the coupling strength, and the oscillators are labeled by $i = 1, \ldots, N \gg 1$. Heterogeneity is modeled via the natural frequencies ω_i, which are drawn from a probability distribution $g(\omega)$. The PRC function Q, measures the degree of advance or delay of the phases when the oscillators are perturbed. We adopt here a PRC with a sinusoidal shape:

$$Q(\theta) = \sin\beta - \sin(\theta + \beta) \qquad (2)$$

We complete the definition of system (1) with the smooth pulse-like signal:

$$P(\theta) = a_n(1 + \cos\theta)^n \qquad (3)$$

Here $n \geq 1$ allows to control the width of the pulses, and a_n is a normalization constant.

Our most important finding[3] is that the Winfree model (1) with the family of PRCs in Eq. (2) belongs to a family of systems that have asymptotic dynamics in a reduced space, called Ott-Antonsen manifold[4]. This important property allows to exactly describe the dynamics of the Winfree model with only two ODEs for Lorentzian

$$g(\omega) = \frac{\Delta/\pi}{(\omega - \omega_0)^2 + \Delta^2} \qquad (4)$$

Using this low-dimensional description, we fully investigate the dynamics of the Winfree model in terms of four parameters: Δ, ε, β, and n controlling the spread of the natural frequencies, the coupling strength, the PRC, and the pulses' width, respectively.

Interestingly, as a result of our study, we find that brief pulse-like interactions generically favor synchronization, what could be conjectured as a reason for their occurrence in nature. This result is not captured using Kuramoto-like models, since these models are only valid for weak coupling and low frequency heterogeneity.

Finally, the potential of our findings is illustrated uncovering a variety of chimera states in networks of pulse-coupled oscillators, which include a new class of chimeras with chaotic dynamics. All in all, we believe our results will foster theoretical advances on the collective dynamics of oscillators' systems, upgrading the mathematical basis of macroscopic synchronization beyond Kuramoto-like models.

---

* ernest.montbrio@upf.edu
† Instituto de Física de Cantabria (CSIC-UC), Santander.

[1] A. T. Winfree, J. Theor. Biol. **16**, 15 (1967).
[2] Y. Kuramoto, in *International Symposium on Mathematical Problems in Theoretical Physics*, Vol. 39 of *Lecture Notes in Physics*, edited by H. Araki (Springer, Berlin, 1975).
[3] D. Pazó and E. Montbrió, *accepted in* Phys. Rev. X.
[4] E. Ott and T. M. Antonsen, Chaos **18**, 037113 (2008)