Formation of localized structures in bistable systems through nonlocal spatial coupling

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In the recent years there has been a lot of interest in the study of extended systems with spatially nonlocal coupling, in the form of an integral over an spatial domain[1–3]. These systems are intermediate between classical evolution equations (e.g. the heat equation) described through Partial Differential Equations and systems of globally coupled constituents. Nonlocal interaction terms appear in chemical and biological systems after adiabatically eliminating a slowly diffusing reactant and also due to density-dependent effects in biological and ecological systems, and in physical systems when long-range interactions are considered.

In this work[4–6] we have studied the effect of nonlocal spatial coupling on the interaction of fronts connecting two equivalent homogeneous stable states. Leveraging spatial dynamics we provide a general framework to understand the effect of the nonlocality on the shape of the fronts connecting two stable states. In particular, for the 1-dimensional real Ginzburg-Landau equation, we show that while for local coupling the fronts are always monotonic and therefore the dynamical behavior leads to coarsening and the annihilation of pairs of fronts, nonlocal terms can induce spatial oscillations in the front tails. The interaction of these oscillatory tails allows for the creation of localized structures, that emerge from pinning between two fronts (a kink and and antikink). In parameter space the region where fronts are oscillatory is limited by three transitions: the modulational instability (MI) of the homogeneous state (a Hamiltonian Hopf (HH) bifurcation), the Belyakov-Devaney (BD) transition in which monotonic fronts acquire spatial oscillations with infinite wavelength, and a crossover in which monotonically decaying fronts develop spatial oscillations with a finite wavelength. We show how these transitions are organized by codimension 2 and 3 points and illustrate how by changing the parameters of the nonlocal coupling it is possible to bring the system into the region where localized structures can be formed.

We discuss the application of three different influence kernels. The first two, Gaussian and mod-exponential, are positive definite and decay exponentially or faster. Both of them exhibit a BD and a MI in the case of repulsive interactions, and LS are found in the region bounded by these two transitions. These transitions can be explained well from a codim-2 point with 4 null eigenvalues. The third kernel, a Mexican-hat kernel, is not positive-definite and exhibits a richer scenario. In particular, in some parameter range it exhibits a BD and a MI, and thus LS, but now for an interaction that overall is attractive, but in which short-range attraction coexists with long-range repulsion. In addition, it exhibits LS in the opposite case, i.e. an interaction is repulsive as a whole, but in which short-range repulsion coexists with long-range attraction. The range of parameter with LS is now bounded by a MI and a crossover line, in which the stability of a pair of real eigenvalues exchanges with a complex quartet, instead of a BD line in which a complex quartet emerges from the collision of two real doublets. This new transition can be also shown to unfold from a different codim-2 point, while the whole scenario results from the unfolding of a codim-3 point with 6 null eigenvalues.

In conclusion, we have studied the emergence of oscillatory tails and LS in a prototypical 1-dimensional model that only supports monotonic tails with a local interaction. We discuss a general theory[5] of the scenario and the application to three different interaction kernels[6]: Gaussian, mod-exponential and Mexican-hat.

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