Firing-rate model for ensembles of quadratic integrate-and-fire neurons

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When studying the collective dynamics of cortical neurons computationally, networks of large numbers of spiking neurons have naturally been the benchmark model. Network models incorporate the most fundamental physiological properties of neurons: sub-threshold voltage dynamics, spiking (via spike generation dynamics or a fixed threshold), and discontinuous synaptic interactions. For this reason, networks of spiking neurons are considered to be biologically realistic.

However, network models of spiking neurons are typically not amenable to analytical work, and thus constitute above all a computational tool. Rather, researchers use reduced or simplified models which describe some measure of the mean activity in a population of cells, oftentimes taken as the firing rate.

Firing-rate models are simple, phenomenological models of neuronal activity, generally in the form of continuous, first-order ordinary differential equations[1]. Such firing-rate models can be analyzed using standard techniques for differential equations, allowing one to understand the qualitative dependence of the dynamics on parameters.

Nonetheless, firing-rate models do not represent, in general, proper mathematical reductions of the original network dynamics but rather are heuristic. As such, there is in general no clear relationship between the parameters in the rate model and those in the full network of spiking neurons.

In this contribution we derive the firing rate description corresponding to a population of heterogeneous quadratic integrate-and-fire neurons (QIF).

An ensemble of i $(i = 1, \ldots, N)$ recurrently coupled QIF writes:

$$\dot{v}_i = v_i^2 + I_{i,\mathrm{ext}} + I_{i,\mathrm{rec}}, \quad \text{if} \;\; v = v_{\mathrm{peak}}, \;\text{then}\; v \to v_{\mathrm{reset}} \tag{1}$$

$$I_{i,\mathrm{ext}} = \eta_i, \tag{2}$$

$$I_{i,\mathrm{rec}} = J_i r\left(t\right) - g_i r\left(t\right)\left(v_i - E\right). \tag{3}$$

Here v_i is the deviation from the voltage at threshold. Additionally, $I_{i,\mathrm{ext}}$ is an external current and $I_{i,\mathrm{rec}}$ is a current due to recurrent connections. The parameters η_i, J_i and g_i are all quenched random variables. The mean field is

$$r\left(t\right) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j}\delta\left(t - t_i^j\right), \tag{4}$$

which corresponds to the sum over all N neurons.

The change of variable $v = \tan(\theta/2)$, allows to transform the QIF model into a phase model —the so-called θ-neuron model—, and to apply the recently discovered Ott-Antonsen[2,3] theory to this problem.

Using this technique, we are able to exactly reduce the ensemble of θ-neurons, to a system of two ordinary differential equations for two macroscopic variables. One of these variables is found to describe the center of the distribution of subthreshold voltages. In addition, the other variable describes the *firing rate* of the population, which corresponds to the width of the distribution of subthreshold voltages. These dynamical equations are then used to investigate the dynamics of the QIF model in full detail.

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